J. Fluid Mech. (1993), *vol.* 246, *pp.* 381–395

The long-wave limit in the asymptotic theory of hypersonic boundary-layer stability

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(Received 25 June 1991 and in revised form 30 March 1992)

This paper discusses the long-wave limit of the asymptotic theory of hypersonic boundary-layer stability for a gas with the Prandtl number $\frac{1}{2} < \sigma < 1$ and with the viscosity–temperature law being a power function. The investigation is confined to the local-parallel approximation.

In the long-wave limit the vorticity mode starts to interact with the acoustic disturbances in the boundary-layer region. The general solution of the linear problem in the boundary-layer inner region is analysed numerically and analytically. This solution is matched with the long-wave vorticity-mode solution near the transition layer. As a result, the inviscid instability problem for a hypersonic boundary layer is formulated. The analytical solution of this problem is found and analysed. The different limits of the solution are considered and the universal forms of the dependence are obtained. A similarity parameter is found which is a function of the Prandtl number and the power in the viscosity–temperature law. A significant change of the solution behaviour is noticed when this parameter passes a critical value. The asymptotic structure of the amplification rate, as a function of the wavenumber, is described and discussed.

1. Introduction

The present study is an extension of Grubin & Trigub (1993, hereinafter referred to as Part 1). All notation coincides with that introduced therein.

The long-wave limit in the asymptotic theory of the hypersonic boundary-layer inviscid instability is investigated. Previous studies by Blackaby, Cowley & Hall (1990) also described in Blackaby, Cowley & Hall (1993) involved the specific case $\sigma = 1, \omega = \frac{1}{2}$. Our study reveals that the results are very sensitive to σ and ω. Moreover, the asymptotic form may be different depending on these values.

One of the objectives of our study was to explain the mysterious sudden decreases and increases in the otherwise smooth $\tilde{c}(\tilde{\alpha})$ dependence obtained in the numerical results of Mack (1969). Another of our objectives was to eliminate an inconsistency in the vorticity-mode asymptotic theory in the limit $\tilde{\alpha} \rightarrow 0$. It was shown in Part 1 that the long-wave limit in the asymptotic problem for the vorticity mode leads to the infinite increase of $c_{\mathrm{tr}}, -c_{\mathrm{ti}}$. This was obviously inconsistent with assumptions used in the problem statement. Principally, two new effects must be taken into account.

The first is because, at $\tilde{\alpha} = O(\epsilon^{1/k_2})$, the wavelength becomes comparable with the boundary-layer thickness. Near the lower edge of the transition layer $\tilde{\pi} \sim \exp(\tilde{\alpha} y_{\mathrm{t}})$ as $y_{\mathrm{t}} \rightarrow -\infty$. Therefore at $\tilde{\alpha} = O(\epsilon^{1/k_2})$ disturbances in the transition layer cannot decay before the boundary layer, where $-y_{\mathrm{t}} = O(\epsilon^{-1/k_2})$, and must be matched with

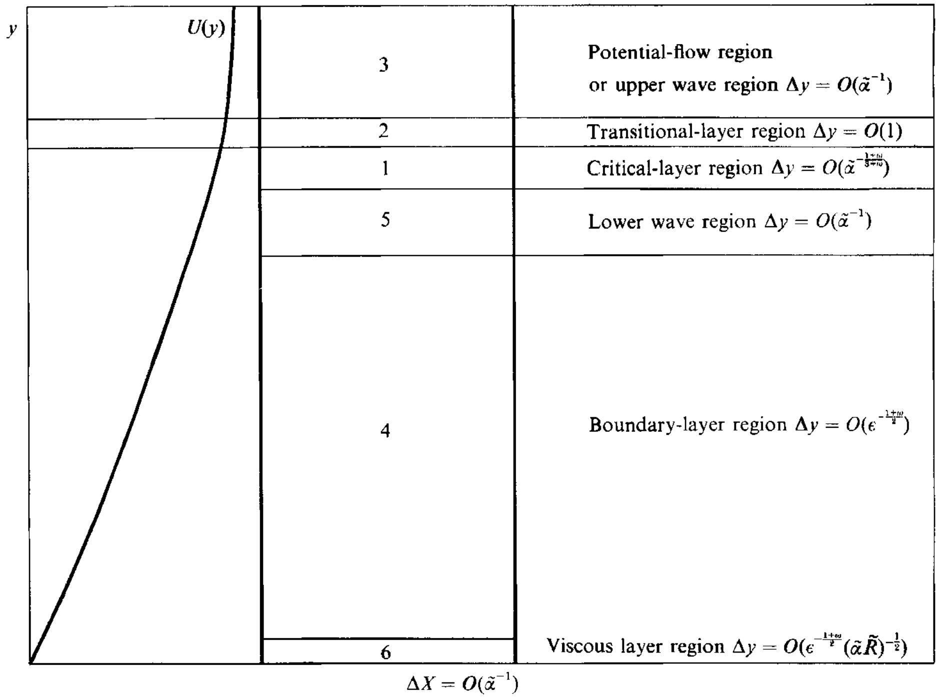

FIGURE 1. The different regions in the flow which emerge as $\tilde{\alpha} \rightarrow 0$.

those in the boundary-layer region. The boundary condition changes – the wave begins to receive information from the near-wall region.

The cause of the second effect is an increase of the external flow Mach number in the system moving with the wave $\hat{M}_\infty = \tilde{M}_\infty(1-\tilde{c})$ as $c_{tr} \rightarrow \infty$. At $c_{tr} \sim \epsilon^{\frac{1}{2}-\nu} C_1^{-1}$, $\hat{M}_\infty = O(1)$ and compressibility effects are not negligible; the term $\tilde{M}^2(\tilde{U}-\tilde{c})^2/T$ in equation (3.4) of Part 1 cannot be discarded.

Taking into account these two effects allows one to determine an asymptotic structure of the $\tilde{c}(\tilde{\alpha})$ dependence as $\epsilon \rightarrow 0$, $\tilde{\alpha} \rightarrow 0$. It will be shown that the dependence is divided into four different regions (see figure 8) as $\tilde{\alpha} \rightarrow 0$. In region 1 the compressibility effect in the potential-flow region 3 in figure 1 plays an important role. This region extends from extremely long wavelength up to region 2 where the wavelength is comparable with the boundary-layer thickness. This part of the curve continues into region 2, but the curve is strongly disturbed there in the narrow regions near the neutral acoustic-mode wavenumbers (regions 3). In every region 3 the local universal structure of the curve consisting of sudden decreases (troughs) and increases (peaks) occur. The width of these narrow regions decreases exponentially as the wavenumber increases. Finally, at $\tilde{\alpha} = O(1)$ we obtain the vorticity-mode region 4 where the maximum growth rate occurs. The asymptotic description produced is in good qualitative agreement with the numerical results of Mack (1969) and predicts some previously unknown effects.

This paper discusses our investigation in five sections as follows. The different regions in the flow which emerge in the asymptotic analysis are shown on figure 1. The equations in the critical-layer region 1 in the long-wave limit are obtained in §2. The behaviour of the disturbances in the transition-layer region 2 is investigated. It is shown that the normal velocity and the pressure do not change in the transition layer. The solution in the potential-flow region 3 is obtained and the conditions on the upper boundary of the transition layer are found. For normal velocity and pressure these conditions coincide with those on the upper boundary of the critical-layer region. Finally, the expansion of the solution near the lower boundary of the critical-layer region is obtained. This expansion contains one arbitrary constant which is to be determined by matching with the lower wave region 5.

Section 3 describes how this constant is obtained by way of an analysis of the general solution of the limiting form of the stability equation in the boundary-layer region 4. The numerical results obtained for helium are presented. The WKB method is used to investigate the problem at large values of the wavenumber. It is shown that the lower wave region 5 is to be considered between the boundary-layer region 4 and the critical-layer region 1. The matching of the solutions in the lower wave region and in the critical-layer region is carried out.

The composite form of the instability problem in the long-wave limit is formulated in §4. The analytical solution of this problem is found and analysed. The different limits 1–4 (mentioned above) are investigated and the universal forms of the $\tilde{c}(\tilde{\alpha})$ dependence at these limits are obtained. It is, surprisingly, observed that these universal forms depend only on the sole similarity parameter $s = (2\sigma-1)/(1+\sigma(1-\omega)/(1+\omega))$. A significant change of the solution behaviour is noticed when the similarity parameter passes the critical value $s=\frac{1}{2}$.

The numerical results for the different values of s are presented in §5. The asymptotic structure of the $\tilde{c}(\tilde{\alpha})$ dependence is described and discussed.

2. The analysis of disturbances near the critical layer

When the wavelength increases, the critical layer moves down to the intermediate region between the boundary and transition layers. The velocity and temperature profiles in the leading-order approximation may be described there by the power dependence, (2.9) of Part 1.

Assume the critical-layer region 1 on figure 1 to be situated at $t \equiv -y_t/\delta = O(1)$, $\delta \to \infty$ as $\tilde{\alpha} \to 0$. From the (2.9) of Part 1 at $t = O(1)$

$$\left.\begin{aligned} &\tilde{U} = 1-\epsilon^{\nu}\delta^{k_1}C_1\bar{C}_1t^{k_1}+\dots, \quad T = \delta^{k_2}\bar{C}_2t^{k_2}+\dots,\\ &\tilde{c} = 1-\epsilon^{\nu}\delta^{k_1}C_1\bar{C}_1\bar{c}, \quad \bar{c}=O(1), \quad \nu=(1+\omega)/2\sigma+\tfrac{1}{2}(1-\omega).\end{aligned}\right\} \tag{2.1}$$

Define the scaling constant so that $\tilde{\pi} = O(1)$ at $t = O(1)$. The dissipative terms of the operator L_ϕ are small compared with the convective ones if $\tilde{R} \gg (\tilde{\alpha}\epsilon^{\nu}\delta^{k_1})^{-1}$. We shall consider only the inviscid limit of the problem in the following analysis and mean this condition to be satisfied. From the condition $L_\phi\tilde{f} \sim \tilde{U}'\tilde{\phi}/T$ it follows that $\tilde{f} \sim \tilde{\phi}/\delta$, and from $L_\phi \tilde{f} \sim \tilde{\pi}/\tilde{M}^2$ it follows that $\tilde{f} \sim \delta^{k_2-k_1}\epsilon^{1-\nu}$. Substituting these estimates into the last of equations (3.1) of Part 1, we have $\tilde{\theta} \sim \delta^{2(k_2-k_1)}\epsilon^{1-2\nu}$.

The scaling factors obtained are used and the following asymptotic representations of the functions at $t = O(1)$ are chosen:

$$\left.\begin{aligned} &\tilde{f} = Qf(t)+\dots, \quad \tilde{\phi} = \mathrm{i}Q\delta\phi(t)+\dots, \quad \tilde{\theta} = \delta^{k_2-k_1}\epsilon^{-\nu}\frac{\bar{C}_2Q}{C_1\bar{C}_1}\theta(t)+\dots,\\ &\tilde{\pi} = 1+O(\tilde{\alpha}^2\delta^2), \quad \tilde{h} = Q\tan\phi h(t), \quad Q = -\delta^{k_2-k_1}\epsilon^{1-\nu}\frac{\bar{C}_2}{C_1\bar{C}_1\gamma\kappa},\\ &\kappa = 2\cos^2\phi/(\gamma-1).\end{aligned}\right\} \tag{2.2}$$

Then we substitute (2.1), (2.2) into (3.1) of Part 1 and find at limit $t = O(1)$, $\epsilon \to 0$, $\tilde{\alpha} \to 0$ the system of equations

$$(\bar{c}-t^{k_1})f+k_1 t^{k_1-1}\phi = t^{k_2}, \quad f-\phi' = 0. \tag{2.3}$$

The condition of constant pressure across the layer is derived from the normal momentum equation. The last equation shows that the disturbances may be considered as incompressible in the critical-layer region – the divergence of their velocity is zero.

The temperature and transverse component of the velocity may be expressed, if ϕ is known,

$$\theta = \frac{k_2 t^{1-k_2\omega}}{\bar{c}-t^{k_1}}\phi, \quad h = \frac{k_1 t^{k_1-1}}{\bar{c}-t^{k_1}}\phi.$$

Two additional assumptions were used at the limit. It was assumed that $\tilde{\alpha}\delta \to 0$, i.e. the wavelength is large compared with the thickness of the region under consideration. This will be proved later when the order of δ is determined. It was also assumed that $\delta^{2k_1-k_2}\epsilon^{2\nu-1} \to 0$, i.e. the relative Mach number $\hat{M} = \tilde{M}_\infty(\tilde{U}-\tilde{c})/T^{\frac{1}{2}}$ tends to zero near the critical layer. This is achieved if the value of $\hat{M}$ in the external flow remains finite as $\epsilon \to 0$, at least for subsonic waves.

In the neighbourhood of $t = 0$ the transition layer is located, where the profiles are not power functions and, therefore, representations (2.1), (2.2) are incorrect. The general solution of (2.3) has the following form as $t \to 0$:

$$\phi = \phi_0(1-t^{k_1}/\bar{c}+\ldots)+t^{k_2+1}/\bar{c}(k_2+1)+\ldots, \tag{2.4}$$

where ϕ_0 is an arbitrary constant. Consequently, the boundary conditions for the system (2.3) may be formulated as

$$f \to 0, \quad \phi \to \phi_0 = \text{const} \quad \text{at} \quad t \to 0. \tag{2.5}$$

The constant ϕ_0 remains indefinite here. It must be determined by matching a solution in the transition layer.

The expansion of the solution at $t \to 0$ allows one to estimate functions in the transition layer ($y_t = O(1)$), $t = O(\delta^{-1})$, (region 2 in figure 1):

$$\tilde{\phi} \sim \delta^{k_2-k_1+1}\epsilon^{1-\nu}, \quad \tilde{\theta} \sim \delta^{k_2-2k_1+1}\epsilon^{1-2\nu}, \quad \tilde{\pi} \sim 1, \quad \tilde{f} \sim \tilde{h} \sim \delta^{k_2-2k_1+1}\epsilon^{1-\nu}.$$

These estimations were substituted in equations (3.1) of Part 1 and the limit $\tilde{\alpha} \to 0, \epsilon \to 0$ in the transition layer was analysed. We found it interesting that viscous effects become significant in the transition and in the critical layers at the same value of $\tilde{R} \sim O(\tilde{\alpha}\epsilon^\nu\delta^{k_1})^{-1}$. Therefore, if the critical layer is inviscid, the transition layer is inviscid too. The normal velocity and pressure were found to be constant in the main approximation:

$$\tilde{\phi} = \mathrm{i}Q\delta\phi_0+\ldots, \quad \tilde{\pi} = 1+\ldots. \tag{2.6}$$

The other functions change and decay exponentially as $y_t \to \infty$:

$$\tilde{f} = \delta^{1-k_1}\frac{Q}{\bar{C}_1\bar{c}}u_t'(y_t)\,\phi_0+\ldots, \quad \tilde{\theta} = -\delta^{1-k_1}\epsilon^{-\nu}\frac{Q}{\bar{C}_1 C_1\bar{c}}T_t'(y_t)\,\phi_0+\ldots,$$

$$\tilde{h} = -\delta^{1-k_1}\frac{Q\tan\phi}{\bar{C}_1\bar{c}}u_t'(y_t)\,\phi_0+\ldots.$$

The value of ϕ_0 may be determined by matching with the solution in the external potential-flow region 3 in figure 1. The thickness of this region is comparable with one wavelength, $y_e = \tilde{\alpha} y_t = O(1)$. The effects of viscosity are negligible there in the leading-order approximation and the problem may be described by equations (3.4), (3.5) of Part 1. The solution is

$$\tilde{\pi} = e^{-g_\infty y_e} + \dots, \quad \tilde{\phi} = -\mathrm{i} g_\infty e^{-g_\infty y_e} [\tilde{\alpha}\gamma \tilde{M}_\infty^2 (1-\tilde{c})]^{-1} + \dots, \tag{2.7}$$

where $g_\infty = [1-\tilde{M}_\infty^2(1-\tilde{c})^2]^{\frac{1}{2}}$, and the square root with a positive real part is chosen, $\mathrm{Re}\,(g_\infty) > 0$. The unique case of neutral supersonic modes, when $\mathrm{Re}\,(g_\infty) = 0$, should be considered separately.

The different limits are further investigated below. Among them there is one at which the compressibility effect in the potential-flow region plays an important role. So, we retain the term $\hat{M}^2$ in the expression for g_∞. If $\delta^{k_1} = o(\epsilon^{\frac{1}{2}-\nu})$, $g_\infty = 1 + \dots$, but if $\delta^{k_1} = O(\epsilon^{\frac{1}{2}-\nu})$, then the complete expression for g_∞ is to be used.

As a result of matching (2.7) with (2.6), we obtain conditions

$$\phi_0 = g_\infty / \bar{c}, \quad \tilde{\alpha}\delta^{k_2+1}\bar{C}_2 = 1. \tag{2.8}$$

From the latter we obtain δ, and thus a set of estimates is completed. Now we can see that the condition $\tilde{\alpha}\delta \to 0$ as $\tilde{\alpha} \to 0$, assumed above, is satisfied.

The general solution of (2.3) at $t \to +\infty$ is

$$\phi = G(t^{k_1} - \bar{c} + O(t^{-k_1})) + t^{k_2-k_1+1}/(2k_1 - k_2 - 1) + O(t^{k_2-2k_1+1}), \tag{2.9}$$

where G is an arbitrary constant.

Constant G is to be determined by matching with a solution in the region situated below the critical layer (region 5 in figure 1). The thickness of this region is comparable with the wavelength. As the wavelength increases, the thickness of the region becomes comparable with the total boundary-layer thickness. We shall consider this case in detail, because it is the most general.

3. The analysis of disturbances in the boundary layer

In the boundary-layer region (4 in figure 1) $y_b = O(1)$; the viscous effects are negligible, except in the near-wall region with a thickness of $\Delta y_b = O((\tilde{\alpha}\tilde{R})^{-\frac{1}{2}})$. Apart from there, for $\tilde{\alpha} \sim \epsilon^{1+\omega}/\tilde{R}$ the wavelength is comparable with Δy_b and viscosity affects the higher acoustic modes. So long as these effects have no influence upon the leading-order approximations considered here, we confine our investigation to the inviscid problem (3.4) of Part 1.

The relative Mach number in the external flow $\hat{M}_\infty$ is considered limited as $\epsilon \to 0$, so that $\tilde{c} = 1 - \epsilon^q c_b, q \geqslant \frac{1}{2}$. In the equation for pressure perturbations we use the variable η_b. As $\epsilon \to 0, \eta_b = O(1)$

$$\tilde{\pi}(\eta) = \bar{\pi}(\eta_b) + O(\epsilon^q), \quad \bar{\pi}'' - 2\frac{u_b'}{u_b}\bar{\pi}' - 2\alpha_b^2 T_b^2 \left(1 - \kappa \frac{u_b^2}{T_b}\right)\bar{\pi} = 0, \tag{3.1}$$

where $\alpha_b = \tilde{\alpha}\epsilon^{-1/\kappa_2}$. In the leading-order approximation a wave moves with the external flow speed; equation (3.1) does not contain c_b.

We choose two linearly independent solutions of (3.1), $\bar{\pi}_A$ and $\bar{\pi}_B$, so that at $\xi \equiv \eta_b - C_0 \to +\infty$

$$\bar{\pi}_A = 1 + O(\xi^{-2(\lambda_2 - 1)}), \qquad \bar{\pi}_B = \frac{\surd 2\lambda_1^2}{\bar{C}_1^2(2\lambda_1+1)}\xi^{-2\lambda_1-1}(1+O(\xi^\beta)); \tag{3.2}$$

the expansion of $\bar{\pi}_A$ does not contain the term $\xi^{-2\lambda_1-1}$. The general solution of the initial-value problem for equation (3.1) with the initial conditions $\bar{\pi}'(0)=0$, $\bar{\pi}(0)=1$ is

$$\bar{\pi} = A(\alpha_b,\kappa)\,\bar{\pi}_A(\eta_b)+B(\alpha_b,\kappa)\,\bar{\pi}_B(\eta_b). \tag{3.3}$$

To obtain the matching conditions for the upper region it is necessary to determine factors A, B, but because of an arbitrary normalization it is enough to know only the ratio B/A. This is done by a numerical analysis of an initial-value problem solution at large values of η_b. Factor A may be obtained easily with high accuracy due to the rapid decay of the function $\bar{\pi}_B$ as $\eta_b\to\infty$. To calculate factor B we need to know all the terms of the power series

$$\bar{\pi}_A = 1+\sum_{j=1}^{\infty} a_j\,\xi^{\theta_j} \tag{3.4}$$

up to $\theta_j<-2\lambda_1-1$. For any λ_1,λ_2 the set of powers $\{\theta_j\}$ must be ordered so that $\theta_j>\theta_{j+1}$. Substituting the expansion (3.4) in (3.1) and using the expansions for u_b, T_b from Appendix A of Part 1 we obtain the factors a_j. To find cumbersome expressions of a_j we used a computer code for symbolic transformations.

The ratio B/A calculated for helium ($\gamma=\frac{5}{3},\sigma=\frac{2}{3},\omega=0.647$) is plotted as a function of α_b in figure 2. Nine terms of the series (3.4) were used in the calculations. Discontinuities are located at points $\alpha_b=\alpha^*_{bn}, n=1,2,\ldots,$ where $A=0,\bar{\pi}\to 0$ at $\eta_b\to\infty$. It was shown by Mack (1969) that the latter is the boundary condition for neutral acoustic modes at the hypersonic limit. In the leading-order approximation the inflexional acoustic modes coincide with the regular ones (those with $\tilde{c}=1$). As α_b increases, the singular regions near the discontinuities become very narrow and almost everywhere B/A takes the form of a continuous monotonically rising function.

When $\alpha_b\to\infty$ an analytical expression for B/A may be obtained by an application of the WKB method (Nayfeh 1973). This method was used to describe higher acoustic modes by Cowley & Hall (1990) and Smith & Brown (1990). Equation (3.1) has one turning point, at $\eta_b=\eta_b^*$, where $q\equiv 2T_b(\kappa u_b^2-T_b)=0$. At $\eta_b<\eta_b^*$ the relative Mach number $\hat{M}>1$ and $q>0$, and the solution of (3.1) oscillates. For $\eta>\eta_b^*$, $\hat{M}>1$ and $q<0$, and the solutions have an exponential nature there.

After the transformation of variables

$$\bar{\pi} = u_b(\eta_b)\,[-z(\eta_b)/q(\eta_b)]^{\frac{1}{4}}v(z), \tag{3.5}$$

$$z=-\left[\tfrac{3}{2}\alpha_b\int_{\eta_b}^{\eta_b^*} q^{\frac{1}{2}}\,\mathrm{d}t\right]^{\frac{2}{3}} \quad \text{for } q>0,$$

$$z=\left[\tfrac{3}{2}\alpha_b\int_{\eta_b^*}^{\eta_b} (-q)^{\frac{1}{2}}\,\mathrm{d}t\right]^{\frac{2}{3}} \quad \text{for} \quad q<0,$$

we obtain the equation for $v(z)$:

$$v''-(z-\chi(\eta_b))\,v=0, \tag{3.6}$$

$$\chi=-\frac{z}{\alpha_b^2 q}\left[\frac{u_b''}{u_b}-2\frac{u_b'^2}{u_b^2}+\frac{5}{16}\frac{q'^2}{q^2}-\frac{1}{4}\frac{q''}{q}-\frac{1}{8}\frac{q'z'}{qz}-\frac{3}{16}\frac{z'^2}{z^2}+\frac{1}{4}\frac{z''}{z}\right].$$

For $\xi=O(1)$, the general solution of (3.6) is

$$v=a\,\mathrm{Ai}\,(z)+b\,\mathrm{Bi}\,(z)+O(\alpha_b^{-2}), \tag{3.7}$$

where Ai, Bi are Airy functions.

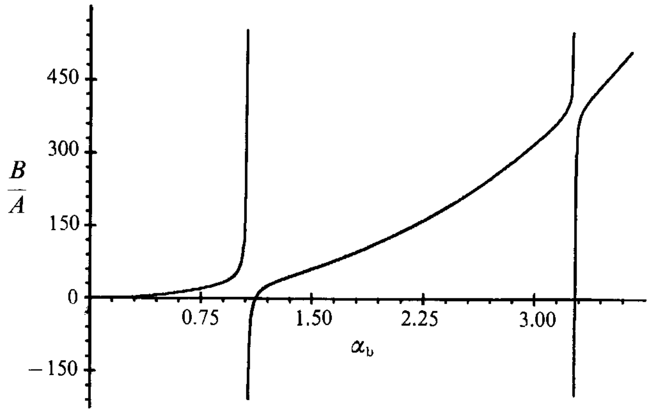

FIGURE 2. The ratio B/A in the general solution of (3.1) obtained numerically for helium, $\psi = 0, T_f = 1$.

The factors a, b are determined from the initial conditions $\bar{\pi}(0) = 1, \bar{\pi}'(0) = 0$:

$$\left.\begin{aligned} a &= \pi^{\frac{1}{2}}(q(0))^{\frac{1}{4}} \sin\left[\alpha_b \tau_0 + \tfrac{1}{4}\pi - Q/\alpha_b + O(\alpha_b^{-2})\right], \\ b &= \pi^{\frac{1}{2}}(q(0))^{\frac{1}{4}} \cos\left[\alpha_b \tau_0 + \tfrac{1}{4}\pi - Q/\alpha_b + O(\alpha_b^{-2})\right], \end{aligned}\right\} \tag{3.8}$$

$$\tau_0 = \int_0^{\eta_*} q^{\frac{1}{2}}\,\mathrm{d}t, \quad Q = \frac{5}{72\tau_0} + \frac{1}{(q(0))^{\frac{1}{2}}}\left[\frac{1}{4}\frac{q'(0)}{q(0)} - u_b'(0)\right].$$

Using the expressions of u_b, T_b from Part 1, it is noticed that $\chi \sim z\xi^{2(\lambda_2-1)}\alpha_b^{-2}$ as $\xi \rightarrow \infty$. Therefore, the approximation (3.5) is invalid at $\xi \sim \alpha_b^{1/(\lambda_2-1)}$, i.e. near the outer edge of boundary layer in the region with thickness comparable with one wavelength (region 5 in figure 1). This region has been defined as 'the lower wave region'. The new variable $t = \sqrt{2C_2}\,\alpha_b\,\xi^{-\lambda_2+1}/(\lambda_2-1)$ is introduced here. From an analysis of (3.1) we find

$$\bar{\pi} = \pi_0(t) + O(\alpha_b^{\beta/(\lambda_2-1)}), \tag{3.9}$$

$$\pi_0'' - (1/t)(2k_1 - k_2)\,\pi_0' - \pi_0 = 0. \tag{3.10}$$

The general solution of (3.10) may be expressed by the modified Bessel functions $I_r, K_r, r = k_1 - \frac{1}{2}k_2 + \frac{1}{2}$,

$$\pi_0 = t^r(C_I I_r(t) + C_K K_r(t)), \tag{3.11}$$

where C_I, C_K are arbitrary constants.

The expansion of (3.7) as $\alpha_b \rightarrow \infty, t = O(1)$ is

$$\begin{aligned} \bar{\pi} = Qt^{k_1-k_2/2}&\left[\tfrac{1}{2}a\,\mathrm{e}^{-\alpha_b\tau_\infty}\,\mathrm{e}^{t}(1+O(\alpha_b^{\beta/(\lambda_2-1)}))\right. \\ &\left.+b\,\mathrm{e}^{\alpha_b\tau_\infty}\,\mathrm{e}^{-t}(1+O(\alpha_b^{\beta/(\lambda_2-1)}))\right](1+O(\alpha_b^{\beta/(\lambda_2-1)})), \end{aligned} \tag{3.12}$$

$$Q = \frac{\lambda_1 C_1}{2^{\frac{1}{4}}(\pi C_2)^{\frac{1}{2}}}\left[\frac{\lambda_2 - 1}{\sqrt{2C_2}\,\alpha_b}\right]^{k_1-k_2/2}, \quad \tau_\infty = \int_{\eta_b^*}^{\infty}(-q)^{\frac{1}{2}}\,\mathrm{d}\xi.$$

The first term in square brackets in (3.12) is exponentially small compared with the second everywhere except in the neighbourhood of the infinite set of points where

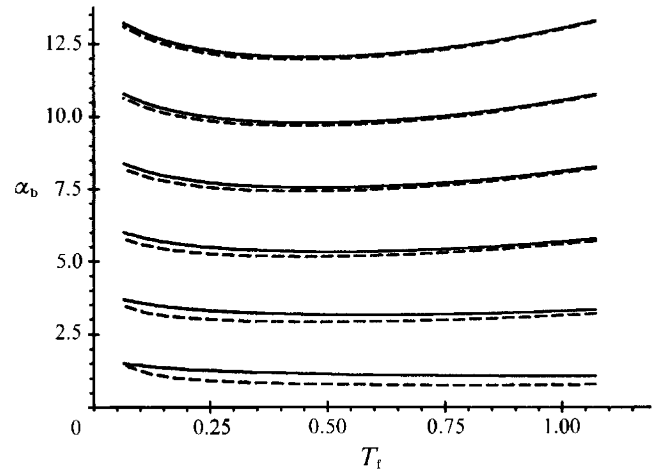

FIGURE 3. The neutral acoustic-mode wavenumbers α^*_{bn} obtained numerically for helium, $\psi = 0$ (——), and their WKB approximation (– – – –), for increasing n.

$b = 0$. At these points $\alpha_b = \alpha_{bn} = \pi(n+\frac{1}{4})/\tau_0 + O(1/n)$, where n is integer, $n \gg 1$. At points α_{bn} where $\pi \to 0$ near the outer boundary, the spectrum α_{bn} approaches the neutral acoustic-mode spectrum α^*_{bn} as $\alpha_b \to \infty$.

General solution (3.11) is matched with (3.12) in the neighbourhood of α_{bn}, where the first and second terms in square brackets in (3.12) are comparable. Using asymptotic expansions of I_r, K_r at $t \to \infty$ (Abramowitz & Stegun 1970), we obtain

$$C_I = (\pi/2)^{\frac{1}{2}} aQ\, e^{-\alpha_b \tau_\infty}, \qquad C_K = (2/\pi)^{\frac{1}{2}} bQ\, e^{\alpha_b \tau_\infty}.$$

Finally, we use asymptotic expansions for I_r, K_r as $t \to 0$ (Abramowitz & Stegun 1970) and compare (3.9) with (3.3) to find

$$\left.\begin{aligned} A &= \frac{\pi C_K}{2^{1-r}\sin(\pi r)\,\Gamma(1-r)}(1+O(\alpha_b^{\beta/(\lambda_2-1)})),\\ B &= \frac{\bar{C}_2\alpha_b^{2r} r}{2^{r-1}\Gamma(1+r)}\left[C_I - \frac{\pi}{2\sin(\pi r)}C_K\right](1+O(\alpha_b^{\beta/(\lambda_2-1)})),\\ \frac{B}{A} &= -\frac{\pi\alpha_b^{2r}\bar{C}_2}{2^{2r-1}\sin(\pi r)\,(\Gamma(r))^2}[1-e^{-2\alpha_b\tau_\infty}\tan(\alpha_b\tau_0+\tfrac{1}{4}\pi)\sin(\pi r)](1+O(\alpha_b^{\beta/(\lambda_2-1)})), \end{aligned}\right\} \tag{3.13}$$

where Γ is the Gamma function.

The representation (3.13) is composite, because the second term is negligible everywhere, except in the neighbourhood of α_{bn}. We reserve this term to analyse the solution in regions 3 of figure 8.

A number of α^*_{bn} from the neutral acoustic-mode spectrum and their WKB approximations α_{bn} are plotted as functions of temperature factor T_f in figure 3. The calculations were performed for helium, $\psi = 0$. The results are in good agreement for $n \geqslant 3$. The curve of B/A dependence outside the singular regions near α^*_{bn} is defined 'the limiting line'. The limiting line is in satisfactory agreement with its WKB approximation only at $\alpha_b \geqslant 50$. We assume is caused by a slow decay of the second and further terms of (3.13) as $\alpha_b \to \infty$.

Since the asymptotic expansion of $\tilde{\pi}$ is known, we are able to obtain the expansion of $\tilde{\phi}$ and perform matching with the functions in the critical-layer region (2.2). This is performed near the points α^*_{bn}, where $B/A \gg 1, G = O(1)$. The final result is

$$\left.\begin{aligned} G &= -\frac{B(\alpha_b)}{A(\alpha_b)}\frac{\epsilon^{(1-s)/(1+s)}}{\alpha_b(\alpha_b \bar{C}_2)^{2/(s+1)}}, \\ s &= \frac{k_2+1}{k_1}-1 = \frac{2\sigma-1}{1+\sigma(1-\omega)/(1+\omega)}. \end{aligned}\right\} \tag{3.14}$$

As seen below, parameter s plays an important role in the long-wave limit. In this study we suppose that $0 < s < 1$.

4. The solution of the inviscid instability problem

The stability problem of the long-wave limit $\epsilon \to 0, \tilde{\alpha} \to 0$ is formulated as:

$$\begin{gathered} (\bar{c}-t^{k_1})\,\phi' + k_1 t^{k_1-1}\phi = t^{k_2}, \\ \phi = g_\infty/\bar{c} \quad \text{at} \quad t = 0, \\ \phi = Gt^{k_1} + \frac{1}{2k_1-k_2-1}t^{k_2-k_1+1} + \ldots \quad \text{at} \quad t \to +\infty. \end{gathered} \tag{4.1}$$

It is necessary to find eigenvalues $\bar{c}$ for which a solution of (4.1) exists with G from (3.14), $g_\infty = (1-\kappa(1-\tilde{c})^2/\epsilon)^{\frac{1}{2}}$, $\tilde{c} = 1-\epsilon^{\nu}\delta^{k_1}C_1\bar{C}_1\bar{c}$, $\tilde{\alpha}\delta^{k_2+1}\bar{C}_2 = 1$.

The problem is composite and may be reformulated in different ways, depending on the correlation of $\tilde{\alpha}$ and $\tilde{\epsilon}$. At the limit $\epsilon \to 0, g_\infty \neq 1$ only at $\tilde{\alpha} \sim \epsilon^{(2\nu-1)(s+1)/2}$, and $G = O(1)$ only near the points α^*_{bn}, otherwise $G = o(1)$.

The solution of (4.1), observing the boundary conditions at $t = 0$, is

$$\phi = (\bar{c}-t^{k_1})\left[\frac{g_\infty}{\bar{c}^2} + \int_0^t \frac{t^{k_2}}{(\bar{c}-t^{k_1})^2}\,\mathrm{d}t\right]. \tag{4.2}$$

Here only the modes are considered for which $0 < \tilde{c}_r < 1$, and a critical layer appears. Without special study we suppose that the well-known relations (Lin 1955) between solutions of the inviscid (4.1) and complete viscous problem are satisfied. This means that the amplified solution of (4.1) obtained by an integration along the real axis t is the limit of the complete problem solution as $\tilde{R} \to \infty$. To obtain the damped solution eigenvalue $\bar{c}_i > 0$, being the limit of the complete problem solution, the integration must be performed in the complex plane t along the contour situated above the pole $t^{k_1} = \bar{c}$.

After an integration along the contours we find

$$\int_0^\infty \frac{t^{k_2}}{(\bar{c}-t^{k_1})^2}\,\mathrm{d}t = -\frac{\pi s}{k_1 \sin(\pi s)}\,\mathrm{e}^{\mp i\pi s}\bar{c}^{s-1} \equiv -K\,\mathrm{e}^{\mp i\pi s}\bar{c}^{s-1}, \tag{4.3}$$

where the minus sign in the exponent is chosen for amplified modes $\frac{3}{2}\pi < \arg(\bar{c}) < 2\pi$, and the plus sign is for damped $0 < \arg(\bar{c}) < \frac{1}{2}\pi$ modes. For the neutral modes both expressions coincide. Demanding that (4.2) satisfies the boundary condition as $t \to +\infty$, we obtain the dispersion relation

$$g_\infty/\bar{c}^2 - K\,\mathrm{e}^{\mp i\pi s}\bar{c}^{s-1} = -G. \tag{4.4}$$

At $\alpha_b = O(1)$ outside the region very close to the points $\alpha_b = \alpha^*_{bn}$ (region 2 on figure 8) it is convenient to represent (4.4) in the following form:

$$\left.\begin{aligned} \alpha_b \frac{g_\infty}{X_b^2} - K\,\mathrm{e}^{\mp \mathrm{i}\pi s}\frac{\alpha_b^2}{X_b^{1-s}} &= \epsilon^{(1-s)/(1+s)}\frac{B(\alpha_b)}{A(\alpha_b)}, \\ \tilde{c} &= 1-\epsilon^{1/(1+s)}C_1\bar{C}_1 X_b, \\ g_\infty = [1-\kappa\epsilon^{(1-s)/(1+s)}C_1^2\bar{C}_1^2 X_b^2]^{\frac{1}{2}}&, \quad X_b = O(1). \end{aligned}\right\} \tag{4.5}$$

A leading-order approximation for the solution of (4.5) as $\epsilon \to 0$ in region 2 is

$$X_b = \frac{\mathrm{e}^{\mathrm{i}\phi}}{(\alpha_b K)^{1/(1+s)}} \equiv X_b^0. \tag{4.6}$$

For the amplified modes only, $\phi = \phi_+ = 2\pi - \pi s/(s+1)$ satisfies the condition $\frac{3}{2}\pi < \phi < 2\pi$. There are no solutions which satisfy the condition $0 < \phi < \frac{1}{2}\pi$ for damped modes.

Near the points $\alpha_b = \alpha^*_{bn}$ (regions 3 in figure 8) the last of equations (4.5) contributes to a leading-order approximation even as $\epsilon \to 0$. The $\tilde{c}(\tilde{\alpha})$ dependence may be expressed in regions 3 in the universal form

$$\left.\begin{aligned} (X_g^{1+s}-1)\,\alpha_g &= X_g^2\,\mathrm{e}^{2\mathrm{i}\phi_+}, \\ X_b &= X_b^0(\alpha^*_{bn})X_g(\alpha_g). \\ \alpha_b &= \alpha^*_{bn}\left[1-\epsilon^{(1-s)/(1+s)}\frac{|X_b^0(\alpha^*_{bn})|^2 B(\alpha^*_{bn})}{\alpha^{*2}_{bn}\mathrm{d}A(\alpha^*_{bn})/\mathrm{d}\alpha_b}\alpha_g\right], \\ \alpha_g &= O(1), \quad X_g = O(1). \end{aligned}\right\} \tag{4.7}$$

It should be noted that solution of (4.7) depends on the sole similarity parameter s.

At $\epsilon^{1/k_2} \ll \tilde{\alpha} \ll 1$ the asymptotic expansion of B/A at $\alpha_b \to \infty$ may be used. In this case we transform (4.5) into

$$\left.\begin{aligned} \frac{g_\infty}{\tilde{X}^2} - K\frac{\mathrm{e}^{\mp \mathrm{i}\pi s}}{\tilde{X}^{1-s}} &= \tilde{\alpha}^{k_2(1-s)/(1+s)}R, \\ \tilde{c} = 1-\epsilon^{\nu}C_1\bar{C}_1\tilde{\alpha}^{-1/(1+s)}\tilde{X}&, \quad R = (B/A)/\alpha_b^{2k_1-k_2+1}. \end{aligned}\right\} \tag{4.8}$$

It is consequence of (3.13) that outside the neighbourhood of $\alpha^*_{bn}, R = O(1)$ as $\alpha_b \to \infty$. Then the last of equations (4.8) does not contribute to the leading-order approximation as $\tilde{\alpha} \to 0$. Therefore, the solution (4.6) remains valid at this limit too.

The limit $\alpha_b = O(\epsilon^{(1-s)/2}), \epsilon \to 0$, when the compressibility effects contribute to the leading approximation ($g_\infty \neq 1$) is the most interesting one (region 1 on the figure 8). The relation (4.5) may be transformed into the universal form again with the sole similarity parameter s:

$$\left.\begin{aligned} (1-X_1^2)^{\frac{1}{2}} &= \alpha_1\,\mathrm{e}^{\mp \mathrm{i}\pi s}X_1^{1+s} + O(\epsilon^{(1-s)/2}\alpha_1 X_1^2), \\ \tilde{c} &= 1-(\epsilon/\kappa)^{\frac{1}{2}}X_1, \quad \alpha_b = \epsilon^{(1-s)/2}(C_1^2\bar{C}_1^2\kappa)^{(1+s)/2}\alpha_1/K, \\ \alpha_1 &= O(1), \quad X_1 = O(1). \end{aligned}\right\} \tag{4.9}$$

The solution (4.6) is the leading-order term of the solution (4.9) as $\alpha_1 \to \infty$. Hence, (4.9) is a leading-order approximation for the dispersion relation (4.5) as $\epsilon \to 0$, $\tilde{\alpha} \to 0$ everywhere except in the neighbourhood of the points α^*_{bn}.

We can see from the analysis of (4.9) that the behaviour of the X_1 dependence in the ranges $0 < s < \frac{1}{2}$ and $\frac{1}{2} < s < 1$ is drastically different. As $\alpha_1 \to 0$ at $0 < s < \frac{1}{2}$, the only solution is an amplified mode with the asymptotic expansion

$$\tilde{c} = 1 - \frac{1}{M_\infty \cos\psi} + \frac{\alpha_1^2}{2M_\infty \cos\psi} \mathrm{e}^{\mathrm{i}2\pi s} + \dots .$$

Therefore, in the neighbourhood of $\alpha_1 = 0$ the waves directed along the flow ($\psi = 0$) are subsonic at $0 < s < \frac{1}{4}$ ($\tilde{c}_r > 1 - 1/M_\infty$) and supersonic at $\frac{1}{4} < s < \frac{1}{2}$.

In the range $\frac{1}{2} < s < 1$ there are no solutions of (4.9), for which $\mathrm{Im}(X_1) \to 0$ as $\alpha_1 \to 0$. Only the supersonic amplified mode with the asymptotic expansion

$$X_1 = (1/\alpha_1^{1/s}) \, \mathrm{e}^{\mathrm{i}\pi(1+1/2s)} + \dots$$

exists there. For this solution the growth rate $\tilde{\alpha}\tilde{c}_i \sim \epsilon^{(3-s)/2+1/k_2}/\alpha_1^{(1-s)/s}$ at fixed ϵ increases as $\tilde{\alpha} \to 0$. This is a very surprising result. The suppositions used to obtain (4.6) are discarded at this limit and to obtain the maximum growth rate we must return to the initial statement. In the present study this investigation was not performed. This limit was resolved for the special case $\sigma = 1, \omega = \frac{1}{2}$ by Blackaby *et al.* (1992).

At points α_{bn}^* the branches of the damped modes begin. An asymptotic expansion of the (4.5) solution near α_{bn}^* is

$$X_b = [-Q(\alpha_b - \alpha_{bn}^*)]^{\frac{1}{2}} + \dots,$$

$$Q = -\frac{\alpha_{bn}^* \, \mathrm{d}A(\alpha_{bn}^*)/\mathrm{d}\alpha_b}{\epsilon^{(1-s)/(1+s)} B(\alpha_{bn}^*)} > 0.$$

If $\alpha_b < \alpha_{bn}^*$, then only neutral modes are possible near α_{bn}^*. But for $\alpha_b > \alpha_{bn}^*$ either amplified or damped modes exist.

5. Discussion

The curves of the dependence of $-\mathrm{Im}(X_1)$ on α_1 obtained from equation (4.9) for amplified modes are represented in figure 4 at different values of the similarity parameter s. As has been established, at $0 < s < \frac{1}{2}$ (figure 4*a*) these lines start from the point $\alpha_1 = 0, \mathrm{Im}(X_1) = 0$. The curves have maxima which are at $0.75 < \alpha_1 < 1.5$ and move to the left as s increases. A form of the curve at $0 < s < \frac{1}{2}$ is in qualitative agreement with the numerical results of Mack (1969). As $s \to \frac{1}{2}$ the curve for the interval $0 < \alpha_1 < \alpha_1^* = \sqrt{2}\, 3^{-\frac{3}{4}} \approx 0.62$ becomes more and more gently sloped, and finally at $s = \frac{1}{2}$ there is a supersonic neutral mode, $\mathrm{Im}(X_1) = 0$. As mentioned above, this is a special case, and in the present study it is considered only as a limit.

At $\frac{1}{2} < s < 1$ (figure 4*b*) the behaviour of the dependence changes significantly, $-\mathrm{Im}(X_1) \to +\infty$ as $\alpha_1 \to 0$. If s is close to $\frac{1}{2}$, a maximum at $\alpha_1 > \alpha_1^*$ and a minimum at $\alpha_1 < \alpha_1^*$ are observed. As s increases, maximum and minimum merges, and the dependence becomes monotonic.

The form of the dependence at $s > \frac{1}{2}$ is unlike the results of calculations of Mack (1969). However, his calculations were performed at finite Mach number ($M_\infty = 10$) for air at wind-tunnel conditions. The dependence of viscosity on temperature in this case is not a power function, but near the upper edge of the boundary layer it may be estimated that approximately $\omega = 1, \sigma = 0.71$, then $s \approx 0.42 < \frac{1}{2}$. As was mentioned, the limit $\epsilon \to 0, \alpha_1 \to 0$ requires further investigation.

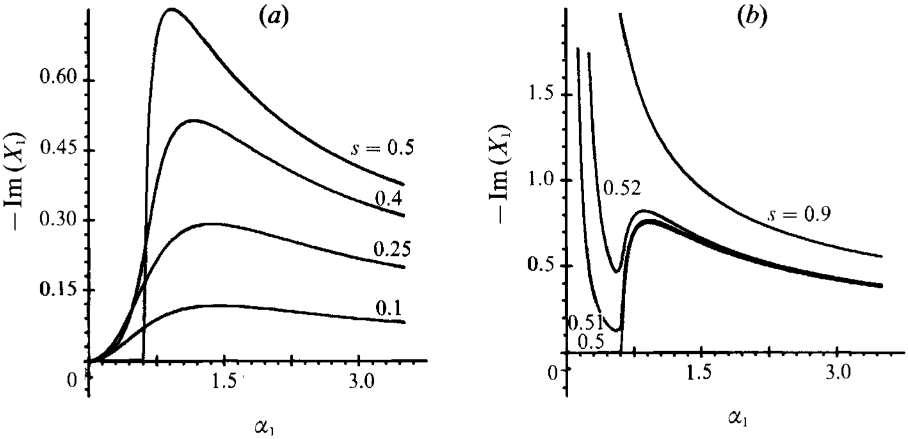

FIGURE 4. The imaginary part of the solution of (4.9) at different values of s (region 1 on figure 8), $\tilde{c} = 1 - X_1/\tilde{M}$.

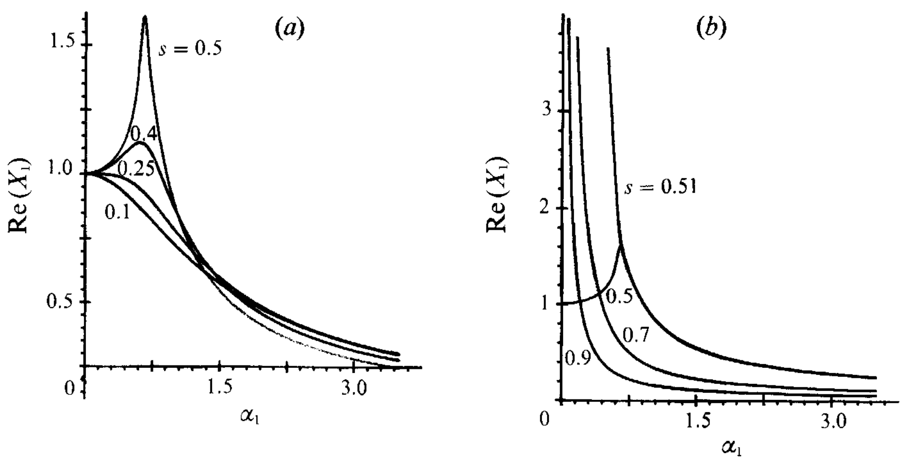

FIGURE 5. The real part of the solution of (4.9) at different values of s. The wave is subsonic at $\mathrm{Re}(X_1) < 1$ and supersonic at $\mathrm{Re}(X_1) > 1$.

The real part of X_1 is plotted as a function of α_1 in figure 5 for a number of values of s. The wave is subsonic at $\mathrm{Re}(X_1) < 1$ and supersonic at $\mathrm{Re}(X_1) > 1$. If $0 < s < \frac{1}{4}$, the wave is supersonic in an interval $0 < \alpha_1 < \alpha_{1s}$, sonic at its edges, and subsonic at $\alpha_1 > \alpha_{1s}$. The α_{1s} increases together with s.

If $s > \frac{1}{2}$ (figure 5*b*), there is also interval $0 < \alpha_1 < \alpha_{1s}$, where the wave is supersonic. Here, however, the wave does not become sonic as $\alpha_1 \rightarrow 0$; $\mathrm{Re}(X_1)$ increases infinitely at this limit. The α_{1s} decreases with the increase of s.

The curves of the universal dependence of $-\mathrm{Im}(X_b/|X_b^0(\alpha_{bn}^*)|)$ on α_g near the points α_{bn}^* obtained from (4.7) for a number of values of s are plotted in figure 6. The dependence has the distinctive form of a sudden trough with a peak at one side of it. The similarity parameter value $s = \frac{1}{2}$ again separates two different kinds of dependence. At $0 < s < \frac{1}{2}$ (figure 6*a*) a peak is situated to the left of a trough, which is in agreement with the numerical results of Mack (1969). If $s = \frac{1}{2}$ the trough proves to be symmetrical, and at $\frac{1}{2} < s < 1$ (figure 6*b*) a peak is located to the right of a trough at $\alpha_g < 0$.

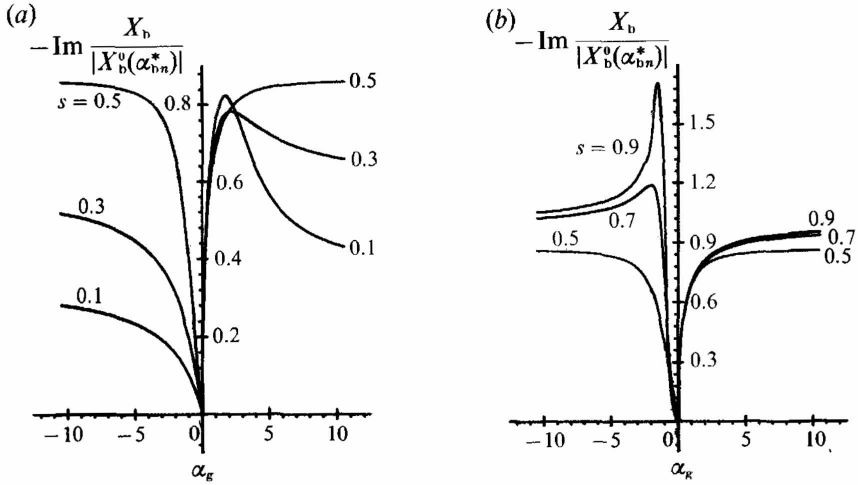

FIGURE 6. The imaginary part of the solution of (4.7) at different values of s (regions 3 in figure 8), $\tilde{c} = 1 - \epsilon^{1/(1+s)} C_1 \bar{C}_1 X_b$.

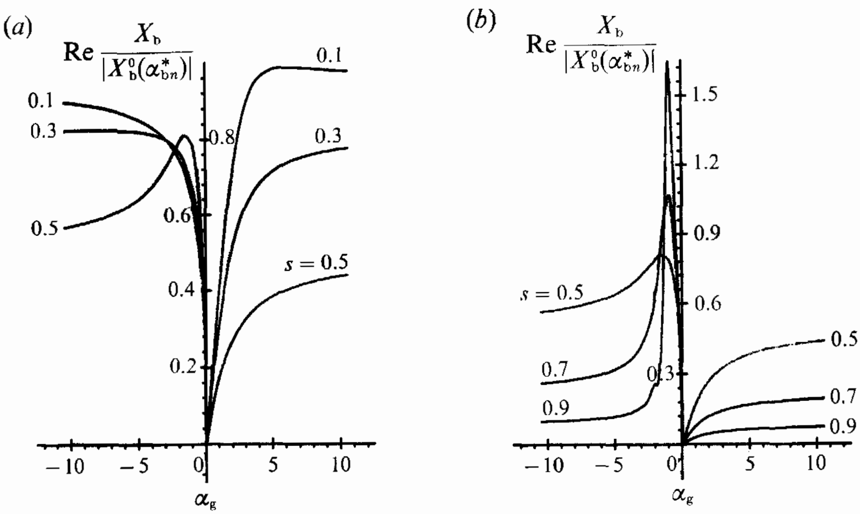

FIGURE 7. The real part of the solution of (4.7).

The curves of the dependence of $\mathrm{Re}\,(X_b/|X_b^0(\alpha_{bn}^*)|)$ on α_g are represented in figure 7(*a, b*). As s increases the asymmetry gains strength and a peak develops to the left of a trough.

The results of Mack (1969) demonstrate that the branches of $\tilde{c}(\tilde{\alpha})$ dependence to the left and to the right of a trough either start from different neutral modes or merge above the neutral line. In the present study we use an approximation in which there is no difference between inflexional and regular neutral modes. Therefore, in this approximation the branches merge in a trough at point $\tilde{c} = 1$ ($X_b = 0$). Close to the point $\alpha_g = 0$ the representation (4.7) is invalid and an internal asymptotic structure of the dependence is revealed there. This structure may be obtained by taking into account the additional terms in the expansion (3.1). More details of the structure of the modes with wavenumbers close to those of acoustic modes, are given for this special case in Blackaby *et al.* (1993).

Now we can assemble the results obtained to describe an asymptotic structure of

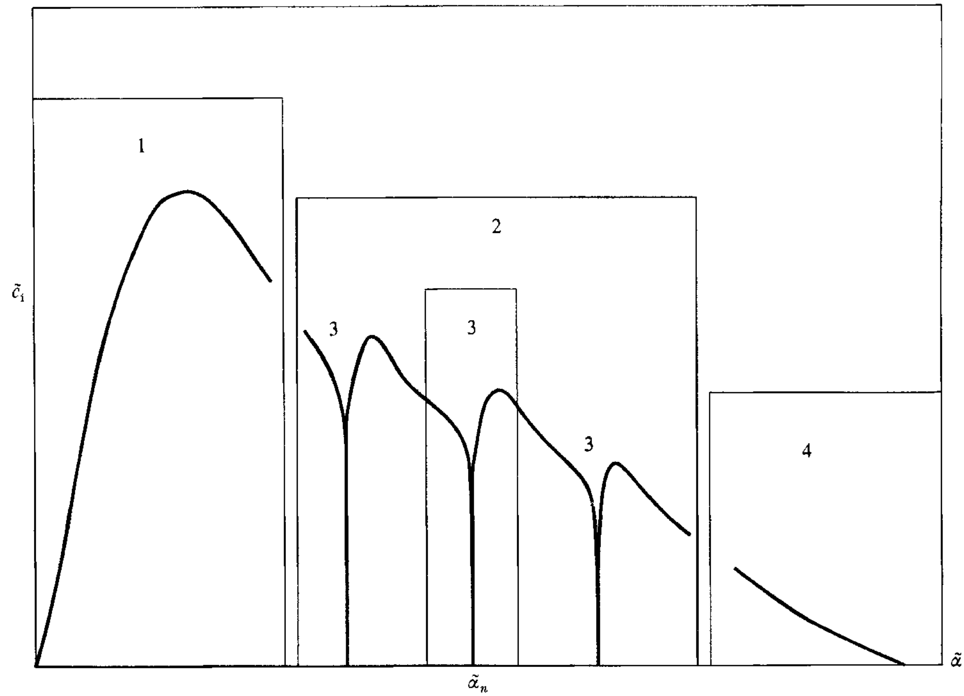

FIGURE 8. The map of the asymptotic forms of high-Mach-number solutions of Rayleigh's instability equation. In region 1, $\tilde{\alpha} \sim (C_1 \cos\psi)\, \epsilon^{1/k_2+(1-s)/2} \sin \pi s$, $\tilde{c}_i \sim \epsilon^{\frac{1}{2}}/\cos\psi$; in region 2, $\tilde{\alpha} \sim \epsilon^{1/k_2}$, $\tilde{c}_i \sim C_1\, \epsilon^{1/(1+s)}$; in regions 3, $\alpha_g \sim 1$, $\tilde{c}_i \sim C_1\, \epsilon^{1/(1+s)}$; in region 4, $\tilde{\alpha} \sim 1$, $\tilde{c}_i \sim C_1\, \epsilon^{\nu}$.

$\tilde{c}_i(\tilde{\alpha})$ dependence as $\epsilon \rightarrow 0$. The range $0 < s < \frac{1}{2}$ only is considered, because at $s > \frac{1}{2}$ further investigations are necessary as $\alpha_1 \rightarrow 0$.

The scheme of the dependence is conventionally represented in figure 8. The range of $\tilde{\alpha}$, where an amplification takes place, divides into four regions. In region 1 the dependence is described by (4.9), $\tilde{c}_i \sim \epsilon^{\frac{1}{2}}/\cos\psi$, $\tilde{\alpha} \sim (C_1 \cos\psi)^{1+s} \epsilon^{1/k_2+(1-s)/2} \sin \pi s$. A compressibility influence is appreciable there. This dependence continues into region 2, where the wavelength and the boundary-layer thickness are comparable, $\tilde{\alpha} \sim \epsilon^{1/k_2}$, $\tilde{c}_i \sim C_1\, \epsilon^{1/(1+s)}$. But there the dependence is considerably disturbed in narrow regions 3 with a width $\Delta\tilde{\alpha} \sim \epsilon^{1/k_2+(1-s)/(1+s)}$, where sudden troughs and peaks occur. These narrow singular regions are the neighbourhood of the points $\tilde{\alpha}^*_{bn}$, corresponding to neutral acoustic modes. The universal structure of the dependence $\tilde{c}(\tilde{\alpha})$ in these regions is described by (4.7).

In region 4 the wavelength is comparable with the transition-layer thickness, $\tilde{\alpha} \sim 1, \tilde{c}_i \sim C_1\, \epsilon^{\nu}$. The region is terminated at the neutral vorticity-mode point $\tilde{\alpha}_n(\sigma, \omega) \sim 1$. This contrasts with the case $\omega = 1$, investigated by Balsa & Goldstein (1990) and Smith & Brown (1990), where $\tilde{\alpha}_n \rightarrow \infty$ as $\epsilon \rightarrow 0$. The maximum amplification rate $\max_{\tilde{\alpha}} (\tilde{\alpha}\tilde{c}_i) \sim C_1\, \epsilon^{\nu}$ is observed in region 4. At $\tilde{\alpha} \sim 1$ the singular regions near the points of neutral acoustic modes exist too but their width is exponentially small.

As Mach number increases $\tilde{c}_i$ decreases, and moreover $\tilde{c}_i \rightarrow 0$ as $\epsilon \rightarrow 0$ for every $\tilde{\alpha}$. In region 1 $\tilde{c}_i$ decreases more slowly and the maximum becomes more distinctive. Regions 1, 2 are displaced to the left, and the width of the troughs decreases.

An increase of the wave inclination angle ψ (we consider only $M_\infty \cos\psi \gg 1$) does not change in the leading order the $\tilde{c}_i(\tilde{\alpha})$ dependence in regions 2, 4. In region 1 $\tilde{c}_i$ increases, and its maximum moves to the left.

Cooling of the wall induces a fast increase of C_1 (Part 1). Consequently, in regions 2, 4 $\tilde{c}_i$ increases and destabilization takes place. In region 1 $\tilde{c}_i$ in the leading-order approximation does not change but the maximum moves to the right. Then at finite ϵ the maximum approaches the first (and the most significant) trough, whose position weakly depends on the temperature factor (figure 3). As a result, the trough 'cuts off' the maximum that we see in the calculation of Mack (1969). Therefore, the cooling in region 1 proves to be stabilizing.

We found the comments and suggestions made by Dr S. Cowley very beneficial and would like to express our appreciation to him. We would also like to thank the INTECO company for its support and assistance in preparing the paper for publishing.

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